Open Access Library Journal
2025, Volume 12, e13315
ISSN Online: 2333-9721
ISSN Print: 2333-9705

Normalized Solutions for a Planar Schrödinger-Poisson System with Inhomogeneous Attractive Interactions

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How to cite this paper: Xue, Q. (2025) Normalized Solutions for a Planar Schrödinger-Poisson System with Inhomogeneous Attractive Interactions. Open Access *Library Journal*, **12**: e13315.
https://doi.org/10.4236/oalib.1113315

Received: March 21, 2025
Accepted: March 28, 2025
Published: April 8, 2025

Abstract

This paper is devoted to the normalized solutions of a planar L^2-critical Schrödinger-Poisson system with an external potential $V(x)=|x|^2$ and inhomogeneous attractive interactions $K(x)\in(0,1)$. Applying the constraint variational method, we prove that the normalized solutions exist if and only if the interaction strength a satisfies $a\in\left(0,a^*\right):=\|Q\|^2_{L^2\left(\mathbb{R}^2\right)}$, where Q is the unique positive solution of $\Delta u-u+u^3=0$ in $\mathbb{R}^2$. Particularly, the refined limiting behavior of positive minimizers is also analyzed as $a\nearrow a^*$.

Subject Areas

Mathematics, Partial Differential Equation

Keywords

Schrödinger-Poisson System, Logarithmic Convolution, Inhomogeneous Attractive Interaction, Normalized Solution

1. Introduction

In this paper, we study the following inhomogeneous elliptic equation with a power potential and a logarithmic convolution potential

$$-\Delta u+\left(|x|^2-\mu\right)u+\left(\ln|x|*u^2\right)u=aK(x)|u|^2u \quad \text{in } \mathbb{R}^2, \tag{1.1}$$

where $\mu\in\mathbb{R}$ is an uncertain Lagrange constant, $a>0$ denotes the strength of attractive interactions, and $K(x)>0$ gives the spatially inhomogeneous attractive interactions. Under the standing wave ansatz $\psi(x,t)=\mathrm{e}^{i\mu t}u(x)$, where i is the imaginary unit, it is well known that (1.1) can be obtained from the time-dependent Schrödinger-Poisson system

$$\begin{cases} i\psi_t - \Delta\psi + |x|^2\psi + \lambda\omega\psi = aK(x)|\psi|^2\psi & \text{in } \mathbb{R}^2\times\mathbb{R}, \\ \Delta\omega = |\psi|^2 & \text{in } \mathbb{R}^2\times\mathbb{R}, \end{cases}$$

where $\psi:\mathbb{R}^2\times\mathbb{R}\to\mathbb{C}$ is the time-dependent wave function and $\lambda\in\mathbb{R}$ is a parameter. The function ω represents an internal potential for a nonlocal self-interaction of the wave function ψ, and the nonlinear term $|\psi|^2\psi$ is frequently used to model the interaction among particles [1]-[3]. In the past several decades, this system, as a cross-disciplinary model bridging quantum mechanics and classical electromagnetism, has garnered great attention owing to its physical relevance. It originates from quantum mechanics [4]-[9] and especially semiconductor physics [10] [11]. We would like to mention the results [12]-[15] for normalized solutions of inhomogeneous elliptic equations, and [15]-[18] with references therein for the Schrödinger-Poisson systems.

In order to investigate the normalized solutions of (1.1), we define the energy functional

$$\begin{aligned} E_a(u) &:= \frac{1}{2}\int_{\mathbb{R}^2}\left[|\nabla u(x)|^2 + |x|^2u^2(x)\right]\mathrm{d}x \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|u^2(x)u^2(y)\,\mathrm{d}x\mathrm{d}y - \frac{a}{4}\int_{\mathbb{R}^2}K(x)u^4(x)\,\mathrm{d}x. \end{aligned} \tag{1.2}$$

Due to the power potential term and the logarithmic convolution term, E_a is not well defined on $H^1(\mathbb{R}^2)$. Stimulated by [14], we consider the space X satisfying

$$X := \left\{u\in H^1(\mathbb{R}^2) : \|u\|_* := \left(\int_{\mathbb{R}^2}|x|^2u^2(x)\,\mathrm{d}x\right)^{\frac{1}{2}} < \infty\right\}$$

with the associated norm

$$\|u\|_X := \left\{\int_{\mathbb{R}^2}\left[|\nabla u|^2 + \left(1+|x|^2\right)u^2(x)\right]\mathrm{d}x\right\}^{\frac{1}{2}},\ u\in X.$$

Recall from ([19], Lemma 3.1) that for any $p\in[2,\infty)$,

$$X \text{ is compactly embedded into } L^p(\mathbb{R}^2). \tag{1.3}$$

As performed in [20], we decompose the logarithmic convolution term as below:

$$F_1(u) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln(1+|x-y|)u^2(x)u^2(y)\,\mathrm{d}x\mathrm{d}y,$$

$$F_2(u) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln\left(1+\frac{1}{|x-y|}\right)u^2(x)u^2(y)\,\mathrm{d}x\mathrm{d}y,$$

and

$$F_1(u) - F_2(u) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|u^2(x)u^2(y)\,\mathrm{d}x\mathrm{d}y.$$

In what follows, we use $\|\cdot\|_p$ to denote the standard Lebesgue norm on $L^p(\mathbb{R}^2)$. Since

$$\ln(1+|x-y|) \le |x-y| \le |x| + |y|,\ x,y\in\mathbb{R}^2,$$

cwe derive from the Hölder inequality that

$$F_1(u) \le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(|x|+|y|)u^2(x)u^2(y)\,\mathrm{d}x\mathrm{d}y \le 2\|u\|_2^3\|u\|_*. \tag{1.4}$$

From the fact that $0<\ln(1+r)<r$ holds for all $r>0$ again, we can deduce from the Hardy-Littlewood-Sobolev inequality (cf. [21]):

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|u(x)||v(y)|}{|x-y|}\mathrm{d}x\mathrm{d}y \le C\|u\|_{\frac{4}{3}}\|u\|_{\frac{4}{3}},\ u,v\in L^{\frac{4}{3}}(\mathbb{R}^2), \tag{1.5}$$

that there exists a constant $C>0$ such that

$$F_2(u) \le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{u^2(x)u^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y \le C\|u\|_{\frac{8}{3}}^4,\ u\in L^{\frac{8}{3}}(\mathbb{R}^2). \tag{1.6}$$

It follows from (1.4) and (1.6) that $\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|u^2(x)u^2(y)\,\mathrm{d}x\mathrm{d}y$ is well defined on X.

Throughout the paper, we assume that the inhomogeneous attractive interactions $K(x)$ satisfy that

$$0<K(x)\le 1 \text{ and } K(0)=\sup_{x\in\mathbb{R}^2}K(x)=1. \tag{1.7}$$

Hence E_a is well defined on X. In the following we focus on studying the minimizers of the constraint variational problem:

$$e(a) := \inf_{u\in S}E_a(u), \tag{1.8}$$

where the manifold S is defined by

$$S := \left\{u\in X : \int_{\mathbb{R}^2}u^2(x)\,\mathrm{d}x = 1\right\}.$$

The main purpose of this paper is to prove the existence, nonexistence and the refined limiting behavior of minimizers for $e(a)$. The proof is closely related to the unique (up to translations) positive solution $Q(x)=Q(|x|)$ of the following elliptic equation (cf. [22] [23]):

$$-\Delta u + u = u^3,\ u\in H^1(\mathbb{R}^2). \tag{1.9}$$

Note from [23] that the function $Q(x)$ satisfies exponential decay in the sense that

$$Q(|x|),|\nabla Q(|x|)| = O\left(|x|^{-\frac{1}{2}}\mathrm{e}^{-|x|}\right) \text{ as } |x|\to\infty. \tag{1.10}$$

In addition, we also need the following Gagliardo-Nirenberg inequality (cf. [24]):

$$\int_{\mathbb{R}^2}|u(x)|^4\mathrm{d}x \le \frac{2}{\|Q\|_2^2}\int_{\mathbb{R}^2}|\nabla u(x)|^2\mathrm{d}x\int_{\mathbb{R}^2}|u(x)|^2\mathrm{d}x,\ u\in H^1(\mathbb{R}^2), \tag{1.11}$$

where the equality is achieved at $u(x)=Q(|x|)$. We can derive from (1.9) and (1.11) that

$$\int_{\mathbb{R}^2}|\nabla Q(x)|^2\mathrm{d}x = \int_{\mathbb{R}^2}Q^2(x)\,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}^2}Q^4(x)\,\mathrm{d}x, \tag{1.12}$$

Applying the above facts, we can establish the existence and nonexistence of minimizers for $e(a)$.

Theorem 1.1. *Let* $Q(x)=Q(|x|)$ *be the unique positive solution of* (1.9) *and* $a^*:=\|Q\|_2^2$.

1) If $a\in(0,a^*)$, then there exists at least one minimizer of $e(a)$;

2) If $a\in(a^*,\infty)$, then there is no minimizer of $e(a)$ and $e(a)=-\infty$;

3) If $a=a^*$ and $1-K(x)=O(|x|^2)$ as $x\to 0$, then there is no minimizer of $e(a)$ and $e(a)=-\infty$.

Moreover, there holds $\lim_{a\nearrow a^*} e(a)=-\infty$ when $1-K(x)=O(|x|^2)$ as $x\to 0$.

Suppose that u_a is a minimizer of $e(a)$ for $a\in(0,a^*)$, then according to the variational theory, u_a satisfies the following Euler-Lagrange equation:

$$-\Delta u_a+|x|^2u_a+\int_{\mathbb{R}^2}\ln|x-y|u_a^2(y)\mathrm{d}y u_a=\mu_a u_a+aK(x)|u_a|^2u_a \quad \text{in } \mathbb{R}^2, \tag{1.13}$$

where $\mu_a\in\mathbb{R}$ denotes the Lagrange multiplier and satisfies

$$\mu_a=2e(a)+\frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|u_a^2(x)u_a^2(y)\mathrm{d}x\mathrm{d}y-\frac{a}{2}\int_{\mathbb{R}^2}K(x)u_a^4\mathrm{d}x. \tag{1.14}$$

That is, u_a is a normalized solution of (1.1). Noticing $E_a(|u_a|)=E_a(u_a)$, we get that $|u_a|$ is also a minimizer of $e(a)$. Together with the strong maximum principle, we next mainly discuss the limiting behavior of positive minimizers.

Theorem 1.2. *Assume that* u_a *is a positive minimizer of* $e(a)$ *for* $a\in(0,a^*)$ *and* $1-K(x)=O(|x|^2)$ *as* $x\to 0$. *Then*

$$\lim_{a\nearrow a^*}2\sqrt{\frac{a^*-a}{a^*}}u_a\left(2\sqrt{\frac{a^*-a}{a^*}}x+x_a\right)=Q(x) \quad \text{in } X,$$

where x_a is the unique global maximum point of u_a as $a\nearrow a^*$.

The proof of Theorem 1.2 requires a series of analysis. We have to overcome the sign-changing property of the logarithmic convolution term. We shall derive the following crucial estimate: there exists a constant $C>0$ such that for any $x\in\mathbb{R}^2$ and for all $a\in(0,a^*)$, there holds

$$\int_{\mathbb{R}^2}\ln\left(1+\frac{1}{|x-y|}\right)v_a^2(y)\mathrm{d}y\le C,$$

where v_a is a suitable scaled function of the minimizer u_a.

We organize the next of this paper as follows. In Section 2, we prove Theorem 1.1 on the existence and nonexistence of minimizers for $e(a)$. In Section 3, we prove Theorem 1.2 on the refined limiting behavior of positive minimizers for $e(a)$ as $a\nearrow a^*$.

2. Existence and Nonexistence of Minimizers

In this section, we shall complete the proof of Theorem 1.1 by applying the Gagliardo-Nirenberg inequalities and the properties of $Q(x)$.

Proof of Theorem 1.1. 1). For any $p\ge 2$ and $u\in H^1(\mathbb{R}^2)$, there results the Gagliardo-Nirenberg inequality (cf. [24]):

$$\|u\|_p \le \left(\frac{p}{2\|Q_p\|_2^{p-2}}\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x\right)^{\frac{p-2}{2p}} \left(\int_{\mathbb{R}^2} |u|^2 \,\mathrm{d}x\right)^{\frac{1}{p}}, \tag{2.1}$$

where Q_p is the positive ground state solution of the following elliptic equation

$$-\frac{p-2}{2}\Delta u + u = u^{p-1}, \quad u \in H^1\left(\mathbb{R}^2\right).$$

By (1.6) and (2.1), we derive that there exists a constant $C > 0$ such that

$$F_2(u) \le C\left(\int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}, \quad u \in S. \tag{2.2}$$

Under the assumption (1.7), we deduce from (1.11) that

$$\int_{\mathbb{R}^2} K(x) u^4 \mathrm{d}x \le \int_{\mathbb{R}^2} u^4 \mathrm{d}x \le \frac{2}{a^*}\int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x, \quad u \in S. \tag{2.3}$$

Notice that $F_1(u) \ge 0$, we infer from (2.2) and (2.3) that for $u \in S$,

$$E_a(u) \ge \left(\frac{1}{2} - \frac{a}{2a^*}\right)\int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^2} |x|^2 u^2 \mathrm{d}x - C\left(\int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}, \tag{2.4}$$

which implies that $E_a(u)$ is bounded from below on S when $a \in \left(0, a^*\right)$.

Letting $\{u_n\} \subset S$ be a minimizing sequence of $e(a)$ for $a \in \left(0, a^*\right)$, we can know from (2.4) that $\int_{\mathbb{R}^2} |\nabla u_n|^2 \,\mathrm{d}x$ and $\int_{\mathbb{R}^2} |x|^2 u_n^2 \mathrm{d}x$ are bounded uniformly with respect to n. Since $\int_{\mathbb{R}^2} u_n^2 \mathrm{d}x = 1$, we then obtain that $\{u_n\}$ is bounded uniformly in X. By (1.3), there exists a function $u \in X$ such that

$$u_n \rightharpoonup u \ \text{ in } X \ \text{ and } \ u_n \to u \ \text{ in } L^p\left(\mathbb{R}^2\right) \ \text{ for } p \in [2, \infty),$$

which implies that

$$\int_{\mathbb{R}^2} u^2 \mathrm{d}x = 1 \ \text{ and } \ \lim_{n\to\infty}\int_{\mathbb{R}^2} K(x) u_n^4 \mathrm{d}x = \int_{\mathbb{R}^2} K(x) u^4 \mathrm{d}x.$$

Then we obtain that $u \in S$. Furthermore, according to ([20], Lemma 2.2), we have

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x-y| u^2(x) u^2(y) \mathrm{d}x\mathrm{d}y \le \liminf_{n\to\infty} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x-y| u_n^2(x) u_n^2(y) \mathrm{d}x\mathrm{d}y.$$

Together with the weak lower semicontinuity of norm, we then deduce from above that

$$e(a) \le E_a(u) \le \liminf_{n\to\infty} E_a(u_n) = e(a),$$

which yields that $E_a(u) = e(a)$. This indicates that u is a minimizer of $e(a)$ for $a \in \left(0, a^*\right)$.

2). Consider the function

$$u_\tau(x) = \frac{\tau}{\sqrt{a^*}} Q(\tau x), \quad \tau > 0.$$

Then $u_\tau \in S$ for all $\tau > 0$. We deduce from (1.12) that

$$\begin{aligned} e(a) \le E_a(u_\tau) = {} & \frac{\tau^2}{4a^*}\int_{\mathbb{R}^2}\left[1 - \frac{a}{a^*}K\left(\frac{x}{\tau}\right)\right] Q^4(x)\mathrm{d}x + \frac{1}{2\tau^2 a^*}\int_{\mathbb{R}^2} |x|^2 Q^2(x)\mathrm{d}x \\ & + \frac{1}{4\left(a^*\right)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x-y| Q^2(x) Q^2(y)\mathrm{d}x\mathrm{d}y - \frac{1}{4}\ln\tau. \end{aligned} \tag{2.5}$$

Through (1.4), (1.10), (2.2) and the assumption (1.7), we obtain from (2.5) that

$$e(a) \le \lim_{\tau\to\infty} E_a(u_\tau) = -\infty \quad \text{for } a > a^*,$$

which implies that there is no minimizer of $e(a)$ and $e(a) = -\infty$ when $a > a^*$.

3). For the case $a = a^*$, we infer from (2.5) that

$$\begin{aligned} e(a) \le E_a(u_\tau) = {} & \frac{\tau^2}{4a^*}\int_{\mathbb{R}^2}\left[1-K\left(\frac{x}{\tau}\right)\right]Q^4(x)\mathrm{d}x + \frac{1}{2\tau^2 a^*}\int_{\mathbb{R}^2}|x|^2 Q^2(x)\mathrm{d}x \\ & + \frac{1}{4(a^*)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|Q^2(x)Q^2(y)\mathrm{d}x\mathrm{d}y - \frac{1}{4}\ln\tau. \end{aligned} \tag{2.6}$$

In virtue of (1.10), we can take $\delta > 0$ satisfying $\int_{|x|>\delta} Q^4\mathrm{d}x < \tau^{-2}$. Hence we get from (1.7) that

$$\frac{\tau^2}{4a^*}\int_{|x|>\delta}\left[1-K\left(\frac{x}{\tau}\right)\right]Q^4(x)\mathrm{d}x \le \frac{1}{2a^*}.$$

By the assumption that $1-K(x) = O\left(|x|^2\right)$ as $x \to 0$, we have

$$\frac{\tau^2}{4a^*}\int_{|x|\le\delta}\left[1-K\left(\frac{x}{\tau}\right)\right]Q^4(x)\mathrm{d}x \le C \quad \text{as } \tau \to \infty.$$

Therefore, we obtain from (1.4), (1.10), (2.2) and (2.6) that

$$e(a^*) \le \lim_{\tau\to\infty} E_{a^*}(u_\tau) = -\infty,$$

which means that there is no minimizer of $e(a^*)$ and $e(a^*) = -\infty$.

In addition, for $a \in (0, a^*)$, choosing $\tau = (a^* - a)^{-\frac{1}{2}}$ in (2.5), we get

$$\begin{aligned} e(a) \le {} & \frac{1}{2a^*} + \frac{\tau^2 a}{4(a^*)^2}\int_{\mathbb{R}^2}\left[1-K\left(\frac{x}{\tau}\right)\right]Q^4(x)\mathrm{d}x + \frac{1}{2\tau^2 a^*}\int_{\mathbb{R}^2}|x|^2 Q^2(x)\mathrm{d}x \\ & + \frac{1}{4(a^*)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|Q^2(x)Q^2(y)\mathrm{d}x\mathrm{d}y - \frac{1}{4}\ln\tau. \end{aligned} \tag{2.7}$$

One can also obtain from the assumption $1-K(x) = O\left(|x|^2\right)$ as $x \to 0$ that

$$\frac{\tau^2 a}{4(a^*)^2}\int_{\mathbb{R}^2}\left[1-K\left(\frac{x}{\tau}\right)\right]Q^4(x)\mathrm{d}x \le C \quad \text{as } a \nearrow a^*.$$

Thus we obtain from (1.4), (1.10), (2.2) and (2.7) that $\lim_{a\nearrow a^*} e(a) = -\infty$. This completes the proof of Theorem 1.1.

3. Limiting Behavior of Minimizers

In this section, we shall prove Theorem 1.2 on the limiting behavior of positive minimizers for $e(a)$ as $a \nearrow a^*$. We first establish some estimates for the positive minimizers of $e(a)$ as $a \nearrow a^*$.

Lemma 3.1. *Assume that* u_a *is a positive minimizer of* $e(a)$ *for* $a \in (0, a^*)$ *and* $1-K(x) = O\left(|x|^2\right)$ *as* $x \to 0$. *Let*

$$\varepsilon_a := \left(\int_{\mathbb{R}^2}|\nabla u_a|^2\mathrm{d}x\right)^{-\frac{1}{2}}, \tag{3.1}$$

$$v_a(x) := \varepsilon_a u_a(\varepsilon_a x + x_a) \ \text{ in } \mathbb{R}^2, \tag{3.2}$$

where x_a is a global maximum point of u_a. Then [(1)]

1) $\varepsilon_a > 0$ satisfies

$$\varepsilon_a \to 0 \ \text{ and } \ \mu_a \varepsilon_a^2 \to -1 \ \text{ as } a \nearrow a^*; \tag{3.3}$$

2) There exists a constant $\eta > 0$, independent of $a \in (0, a^*)$, such that

$$\int_{B_2(0)} v_a^2(x)\mathrm{d}x \ge \eta \ \text{ as } a \nearrow a^*; \tag{3.4}$$

3) v_a satisfies

$$v_a(x) \to \frac{1}{\sqrt{a^*}} Q(|x|) \ \text{ in } H^1(\mathbb{R}^2) \ \text{ as } a \nearrow a^*; \tag{3.5}$$

4) There exist a large constant $R > 0$ and a constant $C > 0$, independent of a, such that

$$|v_a(x)| \le C e^{-\frac{2|x|}{3}} \ \text{ for } |x| \ge R \ \text{ as } a \nearrow a^*. \tag{3.6}$$

Proof. 1). Through (2.2) and (2.3), we have

$$e(a) = E_a(u_a) \ge \frac{a^* - a}{2a^*}\varepsilon_a^{-2} - C\varepsilon_a^{-1} \ge -C\varepsilon_a^{-1}.$$

Together with the fact $\lim\limits_{a \nearrow a^*} e(a) = -\infty$ in Theorem 1.1, we obtain that $\varepsilon_a \to 0$ as $a \nearrow a^*$.

By (3.1), we have

$$\begin{aligned} \varepsilon_a^2 e(a) = &\frac{1}{2} + \frac{\varepsilon_a^2}{2}\int_{\mathbb{R}^2} |x|^2 u_a^2(x)\mathrm{d}x + \frac{\varepsilon_a^2}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(1+|x-y|) u_a^2(x) u_a^2(y)\mathrm{d}x\mathrm{d}y \\ &- \frac{\varepsilon_a^2}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(1+\frac{1}{|x-y|}\right) u_a^2(x) u_a^2(y)\mathrm{d}x\mathrm{d}y - \frac{a\varepsilon_a^2}{4}\int_{\mathbb{R}^2} K(x) u_a^4(x)\mathrm{d}x. \end{aligned} \tag{3.7}$$

Since $\varepsilon_a \to 0$ as $a \nearrow a^*$, we derive from (2.2) that

$$0 \le \varepsilon_a^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(1+\frac{1}{|x-y|}\right) u_a^2(x) u_a^2(y)\mathrm{d}x\mathrm{d}y \le C\varepsilon_a \to 0 \ \text{ as } a \nearrow a^*. \tag{3.8}$$

Note from (2.3) that

$$\frac{1}{2}\left(1 - \frac{a\varepsilon_a^2}{2}\int_{\mathbb{R}^2} K(x) u_a^4(x)\mathrm{d}x\right) \ge 0. \tag{3.9}$$

Hence we deduce from (3.7)-(3.9) that $\liminf\limits_{a \nearrow a^*} \varepsilon_a^2 e(a) \ge 0$. Using the fact that $\lim\limits_{a \nearrow a^*} e(a) = -\infty$ again, we have $\limsup\limits_{a \nearrow a^*} \varepsilon_a^2 e(a) \le 0$. Therefore, we conclude that

$$\lim_{a \nearrow a^*} \varepsilon_a^2 e(a) = 0. \tag{3.10}$$

Furthermore, one can obtain from (3.7)-(3.10) that

$$\lim_{a \nearrow a^*} \varepsilon_a^2 \int_{\mathbb{R}^2} K(x) u_a^4 \mathrm{d}x = \frac{2}{a^*}, \quad \lim_{a \nearrow a^*} \varepsilon_a^2 \int_{\mathbb{R}^2} |x|^2 u_a^2 \mathrm{d}x = 0, \tag{3.11}$$

$$\lim_{a \nearrow a^*} \varepsilon_a^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(1+|x-y|) u_a^2(x) u_a^2(y)\mathrm{d}x\mathrm{d}y = 0. \tag{3.12}$$

Note from (1.14) that

$$\mu_a\varepsilon_a^2 = 2\varepsilon_a^2 e(a) + \frac{\varepsilon_a^2}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x-y| u_a^2(x) u_a^2(y)\,\mathrm{d}x\mathrm{d}y \\ -\frac{a\varepsilon_a^2}{2}\int_{\mathbb{R}^2} K(x) u_a^4(x)\,\mathrm{d}x. \tag{3.13}$$

Together with (3.8) and (3.10)-(3.12), we conclude from (3.13) that $\mu_a\varepsilon_a^2 \to -1$ as $a \nearrow a^*$.

2). Due to (1.13) and (3.2), we see that v_a satisfies

$$-\Delta v_a + \varepsilon_a^2 |\varepsilon_a x + x_a|^2 v_a + \varepsilon_a^2 \left(\int_{\mathbb{R}^2} \ln|x-y| v_a^2(y)\,\mathrm{d}y\right) v_a + \varepsilon_a^2 \ln\varepsilon_a v_a \\ = \varepsilon_a^2 \mu_a v_a + aK(\varepsilon_a x + x_a) v_a^3 \ \text{ in } \mathbb{R}^2. \tag{3.14}$$

We use $\|\cdot\|$ to denote the standard norm on $H^1(\mathbb{R}^2)$. Note that

$$\|v_a\|^2 = \varepsilon_a^2 \int_{\mathbb{R}^2} |\nabla u_a|^2\,\mathrm{d}x + \int_{\mathbb{R}^2} u_a^2\,\mathrm{d}x = 2. \tag{3.15}$$

There exists a constant $C > 0$ such that for any $x \in \mathbb{R}^2$ and $a \in (0, a^*)$,

$$\begin{aligned} &\int_{\mathbb{R}^2} \ln\left(1 + \frac{1}{|x-y|}\right) v_a^2(y)\,\mathrm{d}y \\ &\le \int_{|x-y|<1} \frac{v_a^2(y)}{|x-y|}\,\mathrm{d}y + \int_{|x-y|\ge 1} \frac{v_a^2(y)}{|x-y|}\,\mathrm{d}y \\ &\le \left(\int_{|x-y|<1} \frac{1}{|x-y|^{\frac{3}{2}}}\,\mathrm{d}y\right)^{\frac{2}{3}} \left(\int_{|x-y|<1} v_a^6(y)\,\mathrm{d}y\right)^{\frac{1}{3}} + \int_{|x-y|\ge 1} v_a^2(y)\,\mathrm{d}y \\ &\le C\|v_a\|^2 \le C, \end{aligned} \tag{3.16}$$

Thus we infer from (1.7), (3.3), (3.14) and (3.16) that

$$-\Delta v_a + \frac{5}{9} v_a - a^* v_a^3 \le 0 \ \text{ in } \mathbb{R}^2 \ \text{ as } a \nearrow a^*. \tag{3.17}$$

Since x_a is a maximum point of u_a, the origin is a maximum point of v_a for $a \in (0, a^*)$, which illustrates that $-\Delta v_a(0) \ge 0$ for $a \in (0, a^*)$. Thus we get from (3.17) that there exists some constant $\beta > 0$, independent of a, such that $v_a(0) \ge \beta > 0$ as $a \nearrow a^*$. Applying the De Giorgi-Nash-Moser theory [25], we derive from (3.17) that there exists a constant $C > 0$ such that

$$\left(\int_{B_2(0)} v_a^2\,\mathrm{d}x\right)^{\frac{1}{2}} \ge C \max_{x\in B_1(0)} v_a \ge C\beta := \sqrt{\eta} > 0 \ \text{ as } a \nearrow a^*.$$

3). In view of (3.15), up to a subsequence if necessary, there exists a function $v_0 \in H^1(\mathbb{R}^2)$ such that $v_a \rightharpoonup v_0$ in $H^1(\mathbb{R}^2)$, $v_a \to v_0$ in $L^p_{\mathrm{loc}}(\mathbb{R}^2)$ for $p \in [2, \infty)$, and $v_a \to v_0$ almost everywhere in $\mathbb{R}^2$ as $a \nearrow a^*$. Furthermore, we get $v_0 \not\equiv 0$ from (3.4). Let $o(1)$ denote the infinitesimal quantities as $a \nearrow a^*$. Based on the Brézis-Lieb lemma (cf. [26]), we obtain that as $a \nearrow a^*$,

$$1 = \|v_a\|_2^2 = \|v_0\|_2^2 + \|v_a - v_0\|_2^2 + o(1),$$

$$\left\|K^{\frac{1}{4}}(\varepsilon_a x + x_a) v_a\right\|_4^4 = \left\|K^{\frac{1}{4}}(\varepsilon_a x + x_a) v_0\right\|_4^4 + \left\|K^{\frac{1}{4}}(\varepsilon_a x + x_a)(v_a - v_0)\right\|_4^4 + o(1),$$

and

$$1 = \|\nabla v_a\|_2^2 = \|\nabla v_0\|_2^2 + \|\nabla v_a - \nabla v_0\|_2^2 + o(1).$$

Together with (1.7), (1.11) and (3.11), it yields that

$$\begin{aligned}
0 &= \lim_{a \nearrow a^*} \left(\int_{\mathbb{R}^2} |\nabla v_a|^2 \, dx - \frac{a}{2} \int_{\mathbb{R}^2} K(\varepsilon_a x + x_a) v_a^4 dx \right) \\
&= \int_{\mathbb{R}^2} |\nabla v_0|^2 \, dx - \lim_{a \nearrow a^*} \frac{a}{2} \int_{\mathbb{R}^2} K(\varepsilon_a x + x_a) v_0^4 dx \\
&\quad + \lim_{a \nearrow a^*} \left(\int_{\mathbb{R}^2} |\nabla v_a - \nabla v_0|^2 \, dx - \frac{a}{2} \int_{\mathbb{R}^2} K(\varepsilon_a x + x_a) |v_a - v_0|^4 \, dx \right) \\
&\geq \frac{a^*}{2} \left(\|v_0\|_2^{-2} - 1 \right) \int_{\mathbb{R}^2} v_0^4 dx + \lim_{a \nearrow a^*} \left(\int_{\mathbb{R}^2} |\nabla v_a - \nabla v_0|^2 \, dx - \frac{a}{2} \int_{\mathbb{R}^2} |v_a - v_0|^4 \, dx \right) \\
&\geq \lim_{a \nearrow a^*} \left(1 - \int_{\mathbb{R}^2} |v_a - v_0|^2 \, dx \right) \int_{\mathbb{R}^2} |\nabla v_a - \nabla v_0|^2 \, dx \geq 0.
\end{aligned} \tag{3.18}$$

Therefore, we conclude from (3.18) that

$$\|v_0\|_2 = 1 \text{ and } \int_{\mathbb{R}^2} |\nabla v_a - \nabla v_0|^2 \, dx \to 0 \text{ as } a \nearrow a^*,$$

which further imply that

$$v_a \to v_0 \text{ in } H^1(\mathbb{R}^2) \text{ as } a \nearrow a^*.$$

Additionally, the first equality of (3.18) yields that

$$\int_{\mathbb{R}^2} |\nabla v_0|^2 \, dx = \frac{a^*}{2} \lim_{a \nearrow a^*} K(x_a) \int_{\mathbb{R}^2} v_0^4 dx.$$

Thus we derive from the Lagrange multiplier rule that v_0 satisfies

$$-\Delta v_0 + v_0 - a^* \lim_{a \nearrow a^*} K(x_a) v_0^3 = 0 \text{ in } \mathbb{R}^2.$$

The strong maximum principle implies that $v_0 > 0$ in $\mathbb{R}^2$. Through a simple scaling, the uniqueness (up to translations) of the positive solution of (1.9) ensure that there exists a point $y_0 \in \mathbb{R}^2$ such that

$$v_0(x) = \left(a^* \lim_{a \nearrow a^*} K(x_a) \right)^{-\frac{1}{2}} Q(|x - y_0|).$$

It follows from $\|v_0\|_2 = 1$ that $\lim_{a \nearrow a^*} K(x_a) = 1$. Since the origin is a global maximum point of v_a, it is also a global maximum point of v_0. This indicates that $y_0 = 0$. Hence we get

$$v_a(x) \to \frac{1}{\sqrt{a^*}} Q(|x|) \text{ in } H^1(\mathbb{R}^2) \text{ as } a \nearrow a^*.$$

This convergence is independent of the choice of subsequences and holds true for the whole sequence as well.

4). Using the De Giorgi-Nash-Moser theory (cf. [25]), we derive from (3.15) and (3.17) that

$$\max_{x \in B_1(\xi)} v_a \leq C \left(\int_{B_2(\xi)} v_a^2 dx \right)^{\frac{1}{2}} \text{ for any } \xi \in \mathbb{R}^2,$$

where $C > 0$ is a constant independent of a and ξ. Together with (3.5) and

(3.15), we get that $\{v_a\}$ is bounded uniformly in $L^\infty(\mathbb{R}^2)$ and

$$v_a(x) \to 0 \text{ as } |x| \to \infty \text{ uniformly in } a \nearrow a^*, \tag{3.19}$$

Combining (3.19) with (3.17) then yields that there exists a large constant $R>0$ such that

$$-\Delta v_a + \frac{4}{9} v_a \le 0 \text{ for } |x| \ge R \text{ uniformly in } a \nearrow a^*. \tag{3.20}$$

By applying the comparison principle to (3.20), we can conclude that there exists a positive constant $C>0$ such that

$$|v_a(x)| \le C e^{-\frac{2|x|}{3}} \text{ for } |x| \ge R \text{ uniformly in } a \nearrow a^*. \tag{3.21}$$

Thus we complete the proof of Lemma 3.1.

Now we prove the refined limiting behavior of positive minimizers of $e(a)$ in X as $a \nearrow a^*$.

Proof of Theorem 1.2. Using the exponential decays (1.10) and (3.6), we get that for any $\epsilon > 0$, there exists a constant $R>0$ such that

$$2\int_{B_R^c} |x|^2 v_a^2(x)\mathrm{d}x + 2\int_{B_R^c} |x|^2 \frac{Q^2(x)}{a^*}\mathrm{d}x < \frac{\epsilon}{2}.$$

Combining this with (3.5) we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^2} |x|^2 \left(v_a(x) - \frac{Q(x)}{\sqrt{a^*}} \right)^2 \mathrm{d}x \\
&= \int_{B_R} |x|^2 \left(v_a(x) - \frac{Q(x)}{\sqrt{a^*}} \right)^2 \mathrm{d}x + \int_{B_R^c} |x|^2 \left(v_a(x) - \frac{Q(x)}{\sqrt{a^*}} \right)^2 \mathrm{d}x \\
&\le |R|^2 \int_{B_R} \left(v_a(x) - \frac{Q(x)}{\sqrt{a^*}} \right)^2 \mathrm{d}x + 2\int_{B_R^c} |x|^2 v_a^2(x)\mathrm{d}x + 2\int_{B_R^c} |x|^2 \frac{Q^2(x)}{a^*}\mathrm{d}x \\
&< \epsilon \text{ as } a \nearrow a^*.
\end{aligned}$$

Together with (3.5), we can indicate that $v_a(x) \to \frac{Q}{\sqrt{a^*}}$ in X as $a \nearrow a^*$.

In older to complete the proof of Theorem 1.2, we next need to prove that

$$\varepsilon_a = 2\sqrt{\frac{a^* - a}{a^*}}\left(1 + o(1)\right) \text{ as } a \nearrow a^*. \tag{3.22}$$

Firstly, we derive the upper estimate of $e(a)$ as $a \nearrow a^*$. Setting

$$\tau = \left[\frac{a^*}{4(a^* - a)} \right]^{\frac{1}{2}} > 0$$

into (2.5), we deduce that

$$\begin{aligned}
e(a) \le{}& \frac{1}{8} - \frac{1}{8}\ln a^* + \frac{1}{8}\ln 4(a^* - a) \\
&+ \frac{1}{4(a^*)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x-y| Q^2(x) Q^2(y)\mathrm{d}x\mathrm{d}y + o(1) \text{ as } a \nearrow a^*.
\end{aligned} \tag{3.23}$$

In Addition, using (1.10), (3.5), (3.6) and (3.16), we obtain that

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|v_a^2(x)v_a^2(y)\mathrm{d}x\mathrm{d}y-\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|\frac{Q^2(x)}{a^*}\frac{Q^2(y)}{a^*}\mathrm{d}x\mathrm{d}y\right|\\
&=\left|\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|v_a^2(x)v_a^2(y)\mathrm{d}x\mathrm{d}y-\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|\frac{Q^2(x)}{a^*}v_a^2(y)\mathrm{d}x\mathrm{d}y\right)\right.\\
&\quad\left.-\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|\frac{Q^2(x)}{a^*}\frac{Q^2(y)}{a^*}\mathrm{d}x\mathrm{d}y-\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|\frac{Q^2(x)}{a^*}v_a^2(y)\mathrm{d}x\mathrm{d}y\right)\right|\\
&\le\left|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|\left(v_a^2(x)-\frac{Q^2(x)}{a^*}\right)v_a^2(y)\mathrm{d}x\mathrm{d}y\right|\\
&\quad+\left|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|\frac{Q^2(x)}{a^*}\left(v_a^2(y)-\frac{Q^2(y)}{a^*}\right)\mathrm{d}x\mathrm{d}y\right| \qquad (3.24)\\
&\le\int_{\mathbb{R}^2}|x|\left|v_a^2(x)-\frac{Q^2(x)}{a^*}\right|\mathrm{d}x+\int_{\mathbb{R}^2}\left|v_a^2(x)-\frac{Q^2(x)}{a^*}\right|\mathrm{d}x\int_{\mathbb{R}^2}|y|v_a^2(y)\mathrm{d}y\\
&\quad+\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{v_a^2(y)}{|x-y|}\left|v_a^2(x)-\frac{Q^2(x)}{a^*}\right|\mathrm{d}x\mathrm{d}y\\
&\quad+\int_{\mathbb{R}^2}|x|\frac{Q^2(x)}{a^*}\mathrm{d}x\int_{\mathbb{R}^2}\left|v_a^2(y)-\frac{Q^2(y)}{a^*}\right|\mathrm{d}y+a^*\int_{\mathbb{R}^2}|y|\left|v_a^2(y)-\frac{Q^2(y)}{a^*}\right|\mathrm{d}y\\
&\quad+\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{|x-y|}\frac{Q^2(x)}{a^*}\left|v_a^2(y)-\frac{Q^2(y)}{a^*}\right|\mathrm{d}x\mathrm{d}y\to 0\ \text{ as } a\nearrow a^*.
\end{aligned}
$$

We now give the lower estimate of $e(a)$ as $a\nearrow a^*$. It follows from (1.11), (3.5) and (3.24) that

$$
\begin{aligned}
&e(a)=E(u_a)\\
&=\frac{1}{2}\varepsilon_a^{-2}\left(\int_{\mathbb{R}^2}|\nabla v_a|^2\mathrm{d}x-\frac{a^*}{2}\int_{\mathbb{R}^2}K(\varepsilon_a x+x_a)v_a^4\mathrm{d}x\right)+\frac{1}{2}\int_{\mathbb{R}^2}|\varepsilon_a x+x_a|^2v_a^2(x)\mathrm{d}x\\
&\quad+\frac{1}{4}\ln\varepsilon_a+\frac{1}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|v_a^2(x)v_a^2(y)\mathrm{d}x\mathrm{d}y\\
&\quad+\frac{1}{4}(a^*-a)\varepsilon_a^{-2}\int_{\mathbb{R}^2}K(\varepsilon_a x+x_a)v_a^4\mathrm{d}x\\
&\ge\frac{1}{4}(a^*-a)\varepsilon_a^{-2}\int_{\mathbb{R}^2}v_a^4\mathrm{d}x+\frac{1}{4}\ln\varepsilon_a+\frac{1}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|v_a^2(x)v_a^2(y)\mathrm{d}x\mathrm{d}y \qquad (3.25)\\
&=\frac{a^*-a}{2a^*}(1+o(1))\varepsilon_a^{-2}+\frac{1}{4}\ln\varepsilon_a\\
&\quad+\frac{1}{4(a^*)^2}(1+o(1))\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|Q^2(x)Q^2(y)\mathrm{d}x\mathrm{d}y\\
&\ge\frac{1}{8}+\frac{1}{8}\ln 4(a^*-a)-\frac{1}{8}\ln a^*\\
&\quad+\frac{1}{4(a^*)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|Q^2(x)Q^2(y)\mathrm{d}x\mathrm{d}y+o(1)\ \text{ as } a\nearrow a^*,
\end{aligned}
$$

where the identity in the above inequality is achieved at $\varepsilon_a>0$ satisfying (3.22), *i.e.*, $\varepsilon_a = 2\sqrt{\frac{a^* - a}{a^*}}\left(1 + o(1)\right)$ as $a \nearrow a^*$. We now conclude from (3.23) and (3.25) that

$$e(a) \approx \frac{1}{8} + \frac{1}{8}\ln 4\left(a^* - a\right) - \frac{1}{8}\ln a^* + \frac{1}{4\left(a^*\right)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|Q^2(x)Q^2(y)\mathrm{d}x\mathrm{d}y \quad \text{as } a \nearrow a^*,$$

and $\varepsilon_a > 0$ satisfies (3.22). Moreover, because $v_a \to \frac{Q}{\sqrt{a^*}}$ in X as $a \nearrow a^*$, we obtain from Lemma 3.1 that

$$\lim_{a \nearrow a^*} 2\sqrt{\frac{a^* - a}{a^*}}u_a\left(2\sqrt{\frac{a^* - a}{a^*}}x + x_a\right) = \frac{Q(x)}{\sqrt{a^*}} \quad \text{in } X.$$

This completes the proof of Theorem 1.2.

Through relevant proofs and discussions, the existence of minimizers for $e(a)$ and the refined limiting behavior of positive minimizers for $e(a)$ have been analyzed as $a \nearrow a^*$. These mathematical conclusions provide a theoretical basis for the stability of complex quantum systems and physical phenomena under extreme conditions. In future research, we can discuss the local uniqueness of constraint minimizers as $a \nearrow a^*$ to refine the results.

Conflicts of Interest

The authors declare no conflicts of interest.

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